

Note About Hamiltonian Structure of Non-Linear Massive Gravity

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ABSTRACT: We perform the Hamiltonian analysis of non-linear massive gravity action studied recently in arXiv:1106.3344 [hep-th]. We show that the Hamiltonian constraint is the second class constraint. As a result the theory possesses an odd number of the second class constraints and hence all non physical degrees of freedom cannot be eliminated.

KEYWORDS: Massive Gravity, Hamiltonian Formalism .

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1. Introduction

One of the most challenging problem is to find consistent formulation of massive gravity. The first attempt for construction of this theory is dated to the year 1939 when Fierz and Pauli formulated its version of linear massive gravity [1]¹. However it is very non-trivial task to find a consistent non-linear generalization of given theory and it remains as an intriguing theoretical problem. It is also important to stress that recent discovery of dark energy and associated cosmological constant problem has prompted investigations in the long distance modifications of general relativity, for review, see [3].

Returning to the theories of massive gravity we should mention that these theories suffer from the problem of the ghost instability, for very nice review, see [4]. Since the general relativity is completely constrained system there are four constraint equations along the four general coordinate transformations that enable to eliminate four of the six propagating modes of the metric, where the propagating mode corresponds to a pair of conjugate variables. As a result the number of physical degrees of freedom is equal to two which corresponds to the massless graviton degrees of freedom. On the other hand in case of the massive gravity the diffeomorphism invariance is lost and hence the theory contains six propagating degrees of freedom which only five correspond to the physical polarizations of the massive graviton while the additional mode is ghost.

It is natural to ask the question whether it is possible to construct theory of massive gravity where one of the constraint equation and associated secondary constraint eliminates the propagating scalar mode. It is remarkable that linear Fierz-Pauli theory does not suffer from the presence of such a ghost. On the other hand it was shown by Boulware and Deser [5] that ghosts generically reappears at the non-linear level. However it was shown recently by de Rham and Gobadadze in [14] that it is possible to find such a formulation of the

¹For review, see [2].

massive gravity which is a ghost free in the decoupling limit. Then it was also shown in [15] that this action that was written in the perturbative form can be resummed into fully non-linear actions. It was claimed there that this is the first successful construction of potentially ghost free non-linear actions of massive gravity.

However it is still an open problem whether this theory contains ghost or not, for discussion, see for example [9]. On the other hand S.F. Hassan and R.A. Rosen showed recently in [6] on the non-perturbative level that it is possible to perform such a redefinition of the shift function so that the resulting theory still contains the Hamiltonian constraint. Then it was argued that the presence of this constraint allows to eliminate the scalar mode and hence the resulting theory is the ghost free massive gravity ².

In this paper we again perform the Hamiltonian analysis of the non-linear massive gravity action presented in [7]. The important point which was not addressed in this paper is the character of the Hamiltonian constraint. In fact, the scalar mode can be eliminated on condition that the Poisson brackets between Hamiltonian constraint and all constraints vanish on the constraint surface since then either the time evolution of the Hamiltonian constraint is trivially zero or it induces an additional constraint. The first case corresponds to the situation when the Hamiltonian constraint is the first class constraint while the second case corresponds to the situation when the Hamiltonian constraint together with additional constraint are the second class constraints. However we show that the Poisson brackets between Hamiltonian constraints and also between Hamiltonian constraints and some of other constraints do not vanish at generic points of the constraint surface. In other words the requirement of the preservation of the Hamiltonian constraint fixes corresponding Lagrange multipliers and no new constraints are generated. We should stress that the similar result was derived recently in case of the Hamiltonian analysis of non-projectable version of Hořava-Lifshitz gravity [23], for very nice discussion, see for example [20, 21, 22]. In fact, it was shown there that the Hamiltonian constraint of non-projectable Hořava-Lifshitz gravity is the second class constraint at generic points of the phase space and also that the Hamiltonian strongly vanishes which makes the physical meaning of non-projectable version of Hořava-Lifshitz gravity unclear. In case of the massive gravity the situation is slightly different since the Hamiltonian is not given as the linear combination of constraints and hence does not vanish on the constraint surface. However the fact that the Hamiltonian constraint is the second class means that it is not possible to eliminate all additional physical mode. Moreover, it is not completely clear how to physically interpret the additional 1/2 degree of freedom in the phase space. The structure of this paper is as follows. In the next section (2) we consider the action for the general relativity with the additional term that gives the mass for the graviton when we analyze the perturbative spectrum around the flat space-time. Then we perform the Hamiltonian analysis of given theory and we show that it possesses eight second class constraints. This result is the manifestation of the fact that the diffeomorphism invariance is completely broken which implies that there are no first class constraints. In section (3) we perform the Hamiltonian analysis of the non-linear massive gravity with redefined shift functions. We show that the

²For another works that support the claim that the non-linear gravity action is ghost free, see [26, 27].

momentum conjugate to the lapse function is still the first class constraint however the Hamiltonian constraint is the second class constraint. This result implies that the physical phase space is odd dimensional. In conclusion (4) we outline our result and suggest possible extension of this work. Finally in the Appendix (A) we give an example of the system with the single second class constraint.

2. Massive Gravity

We begin our analysis with the introduction of the notation used in [6, 7]. It is well known that the Fierz-Pauli theory, which is linearized general relativity in flat space is extended by the additional mass term for the metric fluctuations $h_{\mu\nu} = \hat{g}_{\mu\nu} - \eta_{\mu\nu}$

$$\frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h_\mu^\mu h_\nu^\nu) , \quad (2.1)$$

where $\mu, \nu = 0, \dots, 3$ and we use the convention where the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. In order to construct non-linear generalization of Fierz-Pauli theory an additional extra rank two tensor $f_{\mu\nu}$ is introduced. Then the general form of the massive gravity action is [7]

$$S = M_p^2 \int d^4x \sqrt{-\hat{g}} R(\hat{g}) - M_p^2 m^2 \int d^4x \sqrt{-\hat{g}} F(\hat{g}^{-1} f) . \quad (2.2)$$

Note that by definition $\hat{g}^{\mu\nu}$ and $f_{\mu\nu}$ transform under general diffeomorphism transformations $x'^\mu = x'^\mu(x)$ as

$$\hat{g}'^{\mu\nu}(x') = \hat{g}^{\rho\sigma}(x) \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x'^\nu}{\partial x^\sigma} , \quad f'_{\mu\nu}(x') = f_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} . \quad (2.3)$$

It is convenient to parameterize the tensor f_{AB} using four scalar fields ϕ^A and some fixed auxiliary metric $\bar{f}_{\mu\nu}(\phi)$ so that

$$f_{\mu\nu} = \partial_\mu \phi^A \partial_\nu \phi^B \bar{f}_{AB}(\phi) , \quad (2.4)$$

where the metric f_{AB} is invariant under diffeomorphism transformation $x'^\mu = x^\mu(x')$ which however transforms as a tensor under $\phi'^A = \phi'^A(\phi^B)$. The special case $f_{AB} = \eta_{AB}$ corresponds to recently developed Higgs gravity [8, 9, 10, 11, 12]³. In this note we instead consider the unitary gauge fixing of given theory when

$$\phi^A = x^\mu \delta_\mu^A , \quad f_{AB} = \eta_{AB} . \quad (2.5)$$

The gauge fixing (2.5) implies that the action (2.2) is not diffeomorphism invariant and hence there is no gauge freedom left. Finally, the scalar function F that is generally non-linear function of the metric components gives the mass for the graviton when we analyze the fluctuations around the flat space-time.

³For Hamiltonian analysis of given theory, see [13].

In order to find the Hamiltonian formulation of given theory we consider ADM formulation of gravity [18], for review, see [19]. Explicitly, we use 3 + 1 decomposition of the four dimensional metric components

$$\begin{aligned}\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \quad \hat{g}_{0i} = N_i, \quad \hat{g}_{ij} = g_{ij}, \\ \hat{g}^{00} &= -\frac{1}{N^2}, \quad \hat{g}^{0i} = \frac{N^i}{N^2}, \quad \hat{g}^{ij} = g^{ij} - \frac{N^i N^j}{N^2},\end{aligned}\tag{2.6}$$

where $i, j, k, \dots = 1, 2, 3$. Note also that 4-dimensional scalar curvature has following decomposition

$${}^{(4)}R = K_{ij} \mathcal{G}^{ijkl} K_{kl} + {}^{(3)}R, \tag{2.7}$$

where ${}^{(3)}R$ is three-dimensional spatial curvature, K_{ij} is extrinsic curvature defined as

$$K_{ij} = \frac{1}{2N} (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{2.8}$$

where ∇_i is covariant derivative built from the metric components g_{ij} and where \mathcal{G}^{ijkl} is De Witt metric

$$\mathcal{G}^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - g^{ij} g^{kl}. \tag{2.9}$$

Finally note that we omitted terms proportional to the covariant derivatives in (2.7). These terms induce the boundary terms that vanish for suitable chosen boundary conditions. Now for the case when $f_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ the matrix $(\hat{g}f)^\mu_\nu$ takes the form

$$(\hat{g}f)^\mu_\nu = \begin{pmatrix} \frac{1}{N^2} & \frac{N^i}{N^2} \delta_{ij} \\ -\frac{N^i}{N^2} & (g^{ik} - \frac{N^i N^k}{N^2}) \delta_{kj} \end{pmatrix}. \tag{2.10}$$

The presence of this term in the action manifestly breaks the diffeomorphism invariance of given theory. In fact, the action now takes the form

$$S = M_p^2 \int d^4x \sqrt{g} N [K_{ij} \mathcal{G}^{ijkl} K_{kl} + {}^{(3)}R - m^2 F(N, N^i, g_{ij})]. \tag{2.11}$$

From (2.11) it is straightforward to find corresponding Hamiltonian

$$H = \int d^3\mathbf{x} (N \mathcal{H}_T + N^i \mathcal{H}_i), \tag{2.12}$$

where

$$\begin{aligned}\mathcal{H}_T &= \frac{1}{\sqrt{g} M_p^2} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} {}^{(3)}R + \\ &\quad + M_p^2 m^2 \sqrt{g} F(N, N^i, g_{ij}), \\ \mathcal{H}_i &= -2g_{ik} \nabla_j \pi^{jk}.\end{aligned}\tag{2.13}$$

Due to the presence of the mass term this action is highly non-linear in N, N^i which would imply the absence of the first class constraints. Explicitly, since the action (2.11) does not contain the time derivatives of N, N^i we obtain that there are following primary constraints

$$\pi_N \approx 0, \quad \pi_i \approx 0. \quad (2.14)$$

Note that these momenta have following non-zero Poisson brackets with conjugate coordinates

$$\{N(\mathbf{x}), p_N(\mathbf{y})\} = \delta(\mathbf{x} - \mathbf{y}), \quad \{N^i(\mathbf{x}), \pi_j(\mathbf{y})\} = \delta_j^i \delta(\mathbf{x} - \mathbf{y}). \quad (2.15)$$

As the next step we have to check the preservation of the primary constraints (2.14) during the time evolution of the system. Using (2.15) we obtain

$$\begin{aligned} \partial_t \pi_N &= \{\pi_N, H\} = -\mathcal{H}_T - NM_p^2 m^2 \sqrt{g} \frac{\delta F}{\delta N} \equiv -\tilde{\mathcal{H}}_T \approx 0, \\ \partial_t \pi_i &= \{\pi_i, H\} = -\mathcal{H}_i - NM_p^2 m^2 \sqrt{g} \frac{\delta F}{\delta N^i} \equiv -\tilde{\mathcal{H}}_i \approx 0. \end{aligned} \quad (2.16)$$

The crucial point of the massive gravity for the general form of the function F is that the secondary constraints $\tilde{\mathcal{H}}_T \approx 0, \tilde{\mathcal{H}}_i \approx 0$ together with the primary constraints $\pi_N \approx 0, \pi_i \approx 0$ are the second class constraints due to the fact that they depend non-trivially on N, N^i . Explicitly, we have

$$\begin{aligned} \{\pi_N, \tilde{\mathcal{H}}_T\} &= -M_p^2 m^2 \sqrt{g} N \frac{\delta^2 F}{\delta^2 N} \neq 0, \\ \{\pi_i, \tilde{\mathcal{H}}_T\} &= -M_p^2 m^2 \sqrt{g} N \frac{\delta F}{\delta N^i} + NM_p^2 m^2 \sqrt{g} \frac{\delta^2 F}{\delta N \delta N^i} \neq 0, \\ \{\pi_N, \tilde{\mathcal{H}}_i\} &= -M_p^2 m^2 \sqrt{g} \frac{\delta F}{\delta N^i} + NM_p^2 m^2 \sqrt{g} \frac{\delta^2 F}{\delta N \delta N^i} \neq 0, \\ \{\pi_i, \tilde{\mathcal{H}}_j\} &= -NM_p^2 m^2 \sqrt{g} \frac{\delta^2 F}{\delta N^i \delta N^j} \neq 0. \end{aligned} \quad (2.17)$$

As a result we have following collection of the second class constraints $\pi_N, \pi_i, \tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$. The fact that π_N, π_i are the second class constraints means that the conjugate momenta π_N, π_i vanish strongly. Then we solve the constraints $\tilde{\mathcal{H}}_T = 0, \tilde{\mathcal{H}}_i = 0$ for N, N^i that could be expressed as functions of canonical variables g_{ij}, π^{ij} , at least in principle. In other words the dynamical content of theory is given by 6 modes and their conjugate momenta. On the other hand the massive graviton has 5-degrees of freedom so that the additional mode is the well known scalar mode with possible pathological behavior.

3. Hamiltonian Analysis of Non-Linear Massive Gravity

It was argued in [6] that in the specific model of the massive gravity proposed in [7, 15] it is possible to perform suitable redefinition of the shift function N^i in such a way so that the

resulting theory of massive gravity possesses additional constraints. Then it was argued there that the presence of these constraints eliminates the additional scalar mode leaving the physical spectrum of massive gravity only.

Our goal is to perform an explicit Hamiltonian analysis of this theory with redefined shift functions in order to determine its constraint structure. For our purposes it is sufficient to consider following simple model of the non-linear massive gravity action [7, 15]

$$S = M_p^2 \int d^4x \sqrt{g} N [K_{ij} \mathcal{G}^{ijkl} K_{kl} + {}^{(3)}R - 2m^2 (\text{Tr}(\sqrt{\hat{g}^{-1}f} - 3))] , \quad (3.1)$$

where the square root of the matrix is defined such that

$$\left(\sqrt{\hat{g}^{-1}f} \sqrt{\hat{g}^{-1}f} \right)_{\nu}^{\mu} = \hat{g}^{\mu\lambda} f_{\lambda\nu} . \quad (3.2)$$

Following [6] we perform redefinition of the shift function [6]

$$N^i = (\delta_j^i + N D_j^i) n^j \quad (3.3)$$

for new shift functions n^i . Then we demand that the resulting theory is linear in N . In other words, we demand that

$$N(\sqrt{\hat{g}^{-1}f})_{\nu}^{\mu} = \mathbf{A}_{\nu}^{\mu} + N \mathbf{B}_{\nu}^{\mu} , \quad (3.4)$$

where the matrices \mathbf{A}, \mathbf{B} do not depend on N while it can depend on n^i . Then

$$(\hat{g}^{-1}\eta) = \frac{1}{N^2} \mathbf{A}^2 + \frac{1}{N} (\mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{A}) + \mathbf{B}^2 . \quad (3.5)$$

Following [6] we introduce the matrix notation, where n denotes column vector and n^T its transpose. Further, $\eta = \text{diag}(-1, \mathbf{I})$ and $\mathbf{I}_{ij} = \delta_{ij}$, $\mathbf{I}_{ij}^{-1} = \delta^{ij}$. Introducing (3.3) into (3.5) and comparing we find [6]

$$\begin{aligned} \mathbf{A} &= \frac{1}{\sqrt{1 - n^T \mathbf{I} n}} \begin{pmatrix} 1 & n^T \mathbf{I} \\ -n & -n n^T \mathbf{I} \end{pmatrix} , \\ \mathbf{B} &= \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{(g^{-1} - D n n^T D^T) \mathbf{I}} \end{pmatrix} . \end{aligned} \quad (3.6)$$

Inserting (3.3) and (3.6) into (3.1) we obtain following action

$$\begin{aligned} S &= M_p^2 \int d^4x \sqrt{g} N [\tilde{K}_{ij} \mathcal{G}^{ijkl} \tilde{K}_{kl} + \mathbf{D}_{ij} \mathcal{G}^{ijkl} K_{kl} + K_{ij} \mathcal{G}^{ijkl} \mathbf{D}_{kl} + \mathbf{D}_{ij} \mathcal{G}^{ijkl} \mathbf{D}_{kl} + \\ &+ {}^{(3)}R - 2m^2 (\sqrt{1 - n^i \delta_{ij} n^j} + N \text{Tr} \sqrt{g^{-1} \mathbf{I} - D n n^T D^T \mathbf{I}} - 3)] , \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} \mathbf{D}_{ij} &= -\frac{1}{2N} (\nabla_i (N g_{jk} D_l^k n^l) + \nabla_j (N g_{ik} D_l^k n^l)) = \mathbf{D}_{ji} , \\ \tilde{K}_{ij} &= \frac{1}{2N} (\partial_t g_{ij} - \nabla_i n_j - \nabla_j n_i) , \end{aligned} \quad (3.8)$$

and where we used the fact that

$$\mathbf{A}_\mu^\mu = \sqrt{1 - n^i \delta_{ij} n^j} , \quad \mathbf{B}_\mu^\mu = \text{Tr} \sqrt{g^{-1} \mathbf{I} - D n n^T D^T \mathbf{I}} . \quad (3.9)$$

Our goal is to perform the Hamiltonian analysis of the theory defined by the action (3.7). As follows from (3.7) the momenta conjugate to g_{ij} take the form

$$\pi^{ij} = M_p^2 \sqrt{g} \mathcal{G}^{ijkl} (\tilde{K}_{kl} + \mathbf{D}_{kl}) . \quad (3.10)$$

Then it is easy to find the Hamiltonian in the form

$$\begin{aligned} H &= \int d^3 \mathbf{x} \left(N \mathcal{H}_T + \mathcal{H}_j (\delta_i^j + N D_i^j) n^i + 2 M_p^2 m^2 \sqrt{g} \sqrt{1 - n^i \delta_{ij} n^j} \right) , \\ \mathcal{H}_T &= \frac{1}{M_p^2 \sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} {}^{(3)}R + 2 M_p^2 m^2 \sqrt{g} (\text{Tr} \sqrt{(g^{-1} \mathbf{I} - D n n^T D^T \mathbf{I})} - 3) . \end{aligned} \quad (3.11)$$

Due to the absence of the time derivatives of N, n^i in the action (3.7) we see that this theory possesses following collection of the primary constraints

$$\pi_N \approx 0 , \quad \pi_i \approx 0 . \quad (3.12)$$

Now the crucial point is the requirement of the preservation of these primary constraints (3.12) during the time evolution of the system. Explicitly, the requirement of the preservation of the constraints $\pi_N \approx 0, \pi_i \approx 0$ implies

$$\begin{aligned} \partial_t \pi_N &= \{\pi_N, H\} = -\mathcal{H}_T - \mathcal{H}_j D_i^j n^i \equiv -\bar{\mathcal{H}}_T \approx 0 , \\ \partial_t \pi_i &= \{\pi_i, H\} = \left(-\mathcal{H}_j + 2 M_p^2 m^2 \sqrt{g} \frac{\delta_{jk} n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} \right) \left(\delta_i^j + N \frac{\partial(D_k^j n^k)}{\partial n^i} \right) , \end{aligned} \quad (3.13)$$

where we used the fact that

$$\frac{\delta \mathbf{A}_\mu^\mu}{\delta n^i} = -\frac{\delta_{ij} n^j}{\sqrt{1 - n^i \delta_{ij} n^j}} , \quad \frac{\delta \mathbf{B}_\mu^\mu}{\delta n^i} = -\frac{n^k \delta_{kn}}{\sqrt{1 - n^i \delta_{ij} n^j}} \frac{\partial}{\partial n^i} (D^n_p n^p) \quad (3.14)$$

and also the fact that D obeys the equation [6]

$$\sqrt{1 - n^T \mathbf{I} n} D = \sqrt{(g^{-1} - D n n^T D^T) \mathbf{I}} . \quad (3.15)$$

Since the expression $\delta_i^j + N \frac{\partial(D_k^j n^k)}{\partial n^i}$ is the Jacobian of the transformation (3.3) it is non-zero. Then we find that it is natural to introduce following secondary constraint

$$\tilde{\mathcal{H}}_i = \mathcal{H}_i - 2 M_p^2 m^2 \sqrt{g} \frac{\delta_{ij} n^j}{\sqrt{1 - n^i \delta_{ij} n^j}} . \quad (3.16)$$

However using this constraint we observe that the constraint $\bar{\mathcal{H}}_T$ can be written as

$$\bar{\mathcal{H}}_T = \mathcal{H}_T + 2m^2 M_p^2 \sqrt{g} \frac{n^i \delta_{ij} D_k^j n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} + \tilde{\mathcal{H}}_i D^i_j n^j \quad (3.17)$$

and we see that it is natural to introduce new independent constraint $\tilde{\mathcal{H}}_T$ defined as

$$\tilde{\mathcal{H}}_T = \mathcal{H}_T + 2m^2 M_p^2 \sqrt{g} \frac{n^i \delta_{ij} D_k^j n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} . \quad (3.18)$$

The reason why we consider $\tilde{\mathcal{H}}_T$ instead $\bar{\mathcal{H}}_T$ is that the expression \mathcal{H}_i is included in the constraint $\bar{\mathcal{H}}_T$ which makes the calculation of the Poisson brackets between the constraints $\bar{\mathcal{H}}_T$ rather awkward.

Collecting all these results together we find the total Hamiltonian in the form

$$\begin{aligned} H_T = & \int d^3 \mathbf{x} (N \mathcal{H}_T + \mathcal{H}_j (\delta_i^j + D^j_i) n^i + 2M_p^2 m^2 \sqrt{g} \sqrt{1 - n^i \delta_{ij} n^j} + \\ & + v_N \pi_N + v^i \pi_i + u^T \tilde{\mathcal{H}}_T + u^i \tilde{\mathcal{H}}_i) , \end{aligned} \quad (3.19)$$

where v_N, v^i, u^T, u^i are Lagrange multipliers corresponding to the collection of all constraints $\pi_N, \pi_i, \tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$. We see that N appears linearly in the total Hamiltonian (3.19). Finally note that we can express the total Hamiltonian (3.19) using the constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$ as

$$\begin{aligned} H_T = & \int d^3 \mathbf{x} \left[\tilde{u}^T \tilde{\mathcal{H}}_T + \tilde{u}^i \tilde{\mathcal{H}}_i + 2M_p^2 m^2 \sqrt{g} \frac{1}{\sqrt{1 - n^i \delta_{ij} n^j}} + v_N \pi_N + v^i \pi_i \right] \equiv \\ \equiv & \int d^3 \mathbf{x} \left[\mathcal{H}_0 + \tilde{u}^T \tilde{\mathcal{H}}_T + \tilde{u}^i \tilde{\mathcal{H}}_i + v_N \pi_N + v^i \pi_i \right] , \end{aligned} \quad (3.20)$$

where we defined shifted Lagrange multipliers

$$\tilde{u}^T = u^T + N , \quad \tilde{u}^i = n^i + D^i_j n^j + u^i \quad (3.21)$$

and the bare Hamiltonian H_0 as

$$H_0 = 2M_p^2 m^2 \int d^3 \mathbf{x} \sqrt{g} \frac{1}{\sqrt{1 - n^i \delta_{ij} n^j}} . \quad (3.22)$$

To proceed further we have to check the stability of all constraints. To do this we need following Poisson brackets

$$\begin{aligned} \left\{ \pi_N, \tilde{\mathcal{H}}_T \right\} &= 0 , \quad \left\{ \pi_N, \tilde{\mathcal{H}}_i \right\} = 0 , \\ \left\{ \pi_i, \tilde{\mathcal{H}}_T \right\} &= -2m^2 M_p^2 \sqrt{g} \frac{\delta_{ij} D^j_k n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} - 2m^2 M_p^2 \delta_{ij} n^j \frac{n^k \delta_{kl} D^l_m n^m}{(1 - n^i \delta_{ij} n^j)^{3/2}} \equiv \Delta_{\pi_i \tilde{\mathcal{H}}_T} , \\ \left\{ \pi_i, \tilde{\mathcal{H}}_j \right\} &= 2M_p^2 m^2 \sqrt{g} \left(\frac{\delta_{ij}}{\sqrt{1 - n^i \delta_{ij} n^j}} + \frac{\delta_{ik} n^k \delta_{il} n^l}{(1 - n^i \delta_{ij} n^j)^{3/2}} \right) \equiv \Delta_{\pi_i \tilde{\mathcal{H}}_j} . \end{aligned} \quad (3.23)$$

Let us comment these results. First of all we see that the Poisson bracket between π_i and $\tilde{\mathcal{H}}_j$ is non-zero on the whole phase space which implies that π_i and $\tilde{\mathcal{H}}_j$ are the second class constraints. The situation is more complicated in case of the Poisson bracket between π_i and $\tilde{\mathcal{H}}_T$ since this Poisson bracket vanishes on the subspace $n^i = 0$. However this is the isolated point of the measure zero so that we can again say that at the generic point of the phase space π_i and $\tilde{\mathcal{H}}_T$ are the second class constraints.

For further analysis it is convenient to introduce the smeared form of the constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i$

$$\mathbf{T}_T(X) = \int d^3\mathbf{x} X(\mathbf{x}) \tilde{\mathcal{H}}_T(\mathbf{x}) , \quad \mathbf{T}_S(X^i) = \int d^3\mathbf{x} X^i(\mathbf{x}) \tilde{\mathcal{H}}_i(\mathbf{x}) , \quad (3.24)$$

where X, X^i are test functions. Note that in the case of the general relativity we have the constraints

$$\mathcal{H}_T^{GR} = \frac{1}{M_p^2 \sqrt{g}} \pi^{ij} \mathcal{G}_{ijkl} \pi^{kl} - M_p^2 \sqrt{g} {}^{(3)}R , \quad \mathcal{H}_i^{GR} = -g_{ij} \nabla_k \pi^{jk} \quad (3.25)$$

whose smeared forms have following algebra of the Poisson brackets

$$\begin{aligned} \{\mathbf{T}_T^{GR}(X), \mathbf{T}_T^{GR}(Y)\} &= \mathbf{T}_S^{GR}((X\partial_j Y - Y\partial_j X)g^{ji}) , \\ \{\mathbf{T}_S^{GR}(X^i), \mathbf{T}_T^{GR}(Y)\} &= \mathbf{T}_T^{GR}(X^i \partial_i Y) , \\ \{\mathbf{T}_S^{GR}(X^i), \mathbf{T}_S^{GR}(Y^j)\} &= \mathbf{T}_S^{GR}(X^i \partial_i Y^j - Y^j \partial_i X^i) . \end{aligned} \quad (3.26)$$

It is important to stress that the right sides of these Poisson brackets are proportional to the constraints and consequently they vanish on the constraints surface. In other words, the constraints in the general relativity are the first class constraints which is the manifestation of the fact that general relativity is the completely constrained system.

Returning to the case of the non-linear massive gravity we now determine the Poisson brackets between the constraints $\mathbf{T}_T(X), \mathbf{T}_S(X^i)$. Firstly we obtain

$$\begin{aligned} \{\mathbf{T}_T(X), \mathbf{T}_T(Y)\} &= \mathbf{T}_S((X\partial_j Y - Y\partial_j X)g^{ji}) + \\ &+ 2m^2 M_p^2 \int d^3\mathbf{x} (X\partial_i Y - Y\partial_i X) g^{ij} \frac{\sqrt{g} \delta_{jk} n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} \equiv \Delta_{TT}(N, M) . \end{aligned} \quad (3.27)$$

Then we calculate following Poisson bracket

$$\begin{aligned} \{\mathbf{T}_S(X^i), \mathbf{T}_S(Y^j)\} &= \mathbf{T}_S(X^i \partial_i Y^j - Y^j \partial_i X^i) + \\ &+ \int d^3\mathbf{x} (X^i \partial_i Y^j - Y^j \partial_i X^i) \frac{2M_p^2 m^2 \sqrt{g} \delta_{jk} n^k}{\sqrt{1 - n^i \delta_{ij} n^j}} - \\ &- \int d^3\mathbf{x} \partial_k \left[\frac{2M_p^2 m^2 \delta_{ij} n^j}{\sqrt{1 - n^i \delta_{ij} n^j}} \right] (X^k Y^i - X^i Y^k) \\ &\equiv \Delta_{SS}(N^i, M^j) . \end{aligned} \quad (3.28)$$

In the same way we determine the Poisson bracket

$$\{\mathbf{T}_S(X^i), \mathbf{T}_T(Y)\} = \mathbf{T}_T(X^i \partial_i Y) + \Phi_{ST}(n^i, X^i, Y) \equiv \Delta_{ST}(X^i, Y) , \quad (3.29)$$

where the functional $\Phi_{ST}(n^i, g, N^i, M)$ depends on n^i, g_{ij} . Finally we calculate following Poisson brackets

$$\begin{aligned} \{\mathbf{T}_T(X), H_0\} &= -8m^2 \int d^3\mathbf{x} X \frac{\pi^{ij} g_{ji}}{\sqrt{1 - n^i \delta_{ij} n^j}} \equiv \Delta_{TH}(X) \neq 0 , \\ \{\mathbf{T}_S(X^i), H_0\} &= 2m^2 M_p^2 \int d^3\mathbf{x} X^k \partial_k \left(\frac{1}{\sqrt{1 - n^i \delta_{ij} n^j}} \right) \sqrt{g} \equiv \Delta_{SH}(X^i) \neq 0 \end{aligned} \quad (3.30)$$

which are non-zero on the whole phase space.

Now we are ready to analyze the time evolution of the constraints π_N, π_i

$$\begin{aligned} \partial_t \pi_N &= \{\pi_N, H_T\} \approx 0 , \\ \partial_t \pi_i &= \{\pi_i, H_T\} \approx \int d^3\mathbf{x} (u^T \Delta_{\pi_i, \tilde{\mathcal{H}}_T} + u^j \Delta_{\pi_i, \tilde{\mathcal{H}}_j}) = 0 . \end{aligned} \quad (3.31)$$

From the first equation we see that π_N is the first class constraint while the second equation shows that π_i is the second class constraint. On the other hand the time evolution of the constraint $\mathbf{T}_T(X), \mathbf{T}_S(X^i)$ implies

$$\begin{aligned} \partial_t \mathbf{T}_T(X) &= \{\mathbf{T}_T(X), H_T\} = \Delta_{TH}(X) + \Delta_{TT}(N, u^T) + \\ &\quad + \Delta_{TS}(X, u^i) + \Delta_{T, \pi_i}(X, u^i) = 0 , \\ \partial_t \mathbf{T}_S(X^i) &= \{\mathbf{T}_S(X^i), H_T\} = \Delta_{SH}(X^i) + \Delta_{ST}(X, u^i) + \\ &\quad + \Delta_{SS}(X^i, u^j) + \Delta_{S\pi_i}(X^i, v^j) = 0 . \end{aligned} \quad (3.32)$$

For generic situation when $n^i \neq 0$ we have 7 equations (3.31) and (3.32) for unknown 7 Lagrange multipliers u^T, u^i, π^i . In other words we have 7 second class constraints $\tilde{\mathcal{H}}_T, \tilde{\mathcal{H}}_i, \pi_i$ while we have one the first class constraint π_N . The constraints $\tilde{\mathcal{H}}_i = 0, \pi_i = 0$ allow to eliminate n^i, π_i in terms of the phase space variables g_{ij}, π^{ij} . The constraint $\pi_N \approx 0$ can be gauge fixed with the condition $N = 0$ and hence N, π_N are eliminated as well. Finally the second class constraint $\tilde{\mathcal{H}}_T = 0$ eliminates one phase space degree of freedom so that we have 11 physical degrees of freedom. Since the massive gravity should have 10 physical degrees of freedom we see that there is one extra 1/2 degree of freedom whose physical interpretation is unclear. We would like to stress that there is another example of the system with single second class constraint per space-time point which is the chiral boson [24, 25]. The existence of the single second class constraint in given system will be shown in the appendix.

4. Conclusion

In this section we outline our results. We developed the Hamiltonian formalism for non-linear massive gravity in the formulation presented in [6]. We made an emphasis on the careful analysis of the preservation of the constraints during the time evolution of the system. We showed that the Hamiltonian constraint is the second class constraint and hence its time evolution does not generate an additional constraint. As a result this theory possesses the number of degrees of freedom corresponding to the massive gravity together with one extra $1/2$ mode whose physical origin is unclear. In other words we mean that even if the proposal suggested in [6] is very promising it is not sufficient for the complete elimination of all non physical degrees of freedom.

We should also make comments about the relation of our work to the paper [6]. The authors claim that after performing the redefinition of the shift function it is possible to integrate out these shift functions so that we derive the massive gravity action that is function of the physical degrees of freedom only and which is linear in N . Then clearly the requirement of the preservation of the primary constraint $\pi_N \approx 0$ generates the secondary constraint which we denote as Φ . However the crucial point is that the Poisson bracket $\{\Phi(\mathbf{x}), \Phi(\mathbf{y})\}$ cannot be zero or proportional to $\Phi(\mathbf{x})$. In fact, since Φ contains the standard general relativity Hamiltonian constraint together with additional terms we expect that the Poisson brackets $\{\Phi(\mathbf{x}), \Phi(\mathbf{y})\}$ is proportional to $-\nabla_i \pi^{ij}$ and to some additional terms. In case of the general relativity the expression $-\nabla_i \pi^{ij}$ is proportional to the generator of the spatial diffeomorphism which is the constraint as well and hence the algebra of constraints in general relativity is closed. In other words, they are the first class constraints. However in case of the Hamiltonian found [6] there are no such constraints since the shift functions have been integrated out. As a result we mean that it is appropriate to interpret Φ as the second class constraint with all physical consequences.

As the possible extension of our work we mean that it would be certainly very interesting to perform the Hamiltonian analysis of the non-linear massive gravity action written with the help of the Stückelberg fields [17]. We hope to return to this problem in near future, at least in case of the $1+1$ dimensional toy model of the massive gravity action proposed recently in [17].

A. Appendix: Hamiltonian Analysis of Chiral Boson

In this appendix we would like to give an example of the system with single second class constraint. Let us consider the Lagrangian for the scalar field in two dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}(\partial_\sigma \phi)^2 . \quad (\text{A.1})$$

It is easy to find corresponding Hamiltonian

$$H = \frac{1}{2}p_\phi^2 + (\partial_\sigma \phi)^2 . \quad (\text{A.2})$$

Now we subject this theory with the chirality constraint

$$\mathcal{C} = p_\phi - \partial_\sigma \phi = 0 . \quad (\text{A.3})$$

It turns out that this is the second class constraint since

$$\{\mathcal{C}(\sigma), \mathcal{C}(\sigma')\} = -2\partial_\sigma \delta(\sigma - \sigma') . \quad (\text{A.4})$$

Clearly the complete Hamiltonian treatment of given theory consists in the replacement of the Poisson brackets with corresponding Dirac brackets. However the goal of this appendix was to give an explicit example of the well known physical system with the single second class constraint.

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